The core of voting games with externalities

Aymeric Lardon

Submitted to *Games*. Pages 1 - 8.

OPEN ACCESS

Article

The core of voting games with externalities

LARDON Aymeric [1]

[1] University of Saint-Etienne, CNRS UMR 5824 GATE Saint-Etienne-Lyon, 42023 Saint- Etienne, France.

* Author to whom correspondence should be addressed; E-Mail: aymeric.lardon@univ-st-etienne.fr; Tel.: + 33 (0) 477 421 962

Version November 29, 2010 submitted to Games. *Typeset by LaTeX using class file mdpi.cls*

Abstract: The purpose of this article is to analyze a class of voting games in which externalities are present. We consider a society in which coalitions can be formed and where a finite number of voters have to choose among a set of alternatives. A coalition is winning if it can veto any proposed alternative. In our model, the veto power of a coalition is dependent on the coalition formation of the outsiders. We show that whether or not the core is non-empty depends crucially on the expectations of each coalition regarding outsiders' behavior when it wishes to veto an alternative. On the one hand, if each coalition has pessimistic expectations, then the core is non-empty if and only if the dimension of the set of alternatives is equal to one. On the other hand, if each coalition has optimistic expectations, the non-emptiness of the core is not ensured.

Keywords: voting games; externalities; core

1. Introduction

Voting games represent a special class of cooperative games in which some coalitions of voters have the power to enforce their will regardless other voters' actions, while the remaining coalitions are powerless to influence the outcome. A natural concept to predict the outcome of a vote is the core. An alternative belongs to the core if there does not exist some alternative and a winning group of voters in favor of changing the status quo to that alternative. Although voting games constitute a rather special class of cooperative games, the existence of a core alternative is by no mean ensured: the famous "paradox of voting" is an elementary three-player example for which the core is empty.

The classification of voting games according to the emptiness or non-emptiness of the core has been the object of a wide literature. Consider a society in which coalitions can be formed and where a finite number, $n \in \mathbb{N}$, of voters have to choose an alternative from a set in the p-dimensional Euclidean space $\mathbb{R}^p$. Assuming that the preference relation over the set of alternatives of each voter is continuous and convex, Greenberg [2] showed that if the set of winning coalitions of a voting game consists of all coalitions with more than $pn/(p+1)$ individuals then the core is non-empty. In the same way, Nakamura [5] determined the upper bound on the cardinality of the set of alternatives of a voting game, called the Nakamura number, which would guarantee the existence of a core alternative. The Nakamura number is defined as the size of the smallest collection of winning coalitions having an empty intersection. If such a collection does not exist, then the Nakamura number is equal to infinity. So, the non-emptiness of the core is directly related to the combinatorial structure of the winning coalitions. Schofield [6], Strnad [7] and Le Breton [4] have generalized Greenberg's result [2] to arbitrary voting games. More precisely, they prove that if p is smaller than or equal to the Nakamura number of a voting game minus two, and if individual preference relations are continuous and convex then the core is non-empty. Otherwise, if p is strictly greater than the Nakamura number of a voting game minus two, there exists a profile of preference relations for which the core is empty.

Coalition formation games with externalities were first modeled by Thrall and Lucas [8] as partition function games. For this class of games, Hafalir [3] shows that a convexity property ensures the non-emptiness of the core with singletons and pessimistic expectations.[1] Funaki and Yamato [1] study the core of an economy with a common pool resource by means of partition function games. On the one hand, if each coalition has pessimistic expectations, then the core is non-empty. On the other hand, if it has optimistic expectations, the core is empty when there are more than four players.

In continuation of these works, we study the class of voting games with externalities. In particular, we investigate whether it is possible to guarantee the non-emptiness of the core when each coalition is embedded into a partition. A coalition embedded into a partition is winning if it can veto any proposed alternative. Contrary to voting games without externalities, a coalition can be winning for a partition and non-winning for another partition. In other words, whether or not a coalition is winning depends on the coalition formation of the outsiders, and *de facto*, the non-emptiness of the core too.

On the one hand, assume that the members of a coalition pessimistically expect that the outsiders react in the worst possible way for them when they wish to veto an alternative. In the present model, this will correspond to the case where outsiders will form the largest possible coalition. We prove that the core is non-empty if and only if the dimension of the set of alternatives is equal to one.
On the other hand, assume that the members of a coalition optimistically expects that the outsiders react in the best way for them when they wish to veto an alternative. In the present model, this will correspond to the case where each outsider will form a singleton. We show that there always exists a profile of preference relations for which the core is empty. Therefore, whether or not the core is non-empty depends crucially on the expectations of each coalition about outsiders' behavior when it wishes to veto an alternative.

This article is organized as follows. Section 2 describes voting games with externalities. In section 3 we prove that the core is non-empty when each coalition has a pessimistic view regarding the coali-

[1] We refer to Hafalir [3] for the definition of the core of partition function games.

tion formation of the outsiders. In section 4, we show that there exists a profile of preference relations for which the core is empty when expectations of each coalition are optimistic regarding the coalition formation of the outsiders. Section 5 gives some concluding remarks.

2. The model

The society consists of a set of voters $N = \{1, \ldots, n\}$ where $n \geq 3$. The voters have to choose an alternative from a nonempty, convex and compact set $X \subseteq \mathbb{R}^p$, $p \in \mathbb{N}$. Each voter $i \in N$ has preferences represented by a complete preorder $\succeq_i$ over X. We denote by $\mathcal{P}$ the set of complete preorders over X. A profile of preference relations is an element $\succeq = (\succeq_1, \ldots, \succeq_n) \in \mathcal{P}^n$. We denote by $\mathcal{P}_{co} \subseteq \mathcal{P}$ the set of continuous,[2] convex[3] and complete preorders over X.

We denote by $S(N)$ and $\Pi(N)$ the sets of all non-empty coalitions and all partitions of N respectively. Let S and ρ be the representative elements of $S(N)$ and $\Pi(N)$ respectively. We denote by $S_\rho(i)$ the unique coalition $S \in \rho$ containing voter i. Let W^e be a correspondence defined from $\Pi(N)$ to $S(N)$ that associates to each partition $\rho \in \Pi(N)$ a (possibly empty) set of winning coalitions such that $W^e(\rho) \subseteq \rho$. Throughout this article, we assume that W^e satisfies the two following conditions:

- **Properly (P)**: for any $\rho \in \Pi(N)$, $S \in W^e(\rho)$ implies $T \notin W^e(\rho)$ for each $T \in \rho$ where $T \neq S$;
- **Splitting of Outsiders (SO)**: for any $\rho \in \Pi(N)$ and any $\rho' \in \Pi(N)$ such that $S \in \rho \cap \rho'$ and $S_{\rho'}(i) \subseteq S_\rho(i)$ for each $i \in N \backslash S$, $S \in W^e(\rho)$ implies $S \in W^e(\rho')$.

Condition (P) means that any partition $\rho \in \Pi(N)$ admits at most one winning coalition while condition (SO) specifies that a winning coalition $S \in S(N)$ for a partition $\rho \in \Pi(N)$ is still winning when the other coalitions embedded into ρ split into smaller groups.

Definition 1 *A voting game (with externalities) is a pair (N, W^e) where N is a set of voters and W^e is a correspondence of winning coalitions.*

As discussed in the introduction, the core seems to be a natural concept to predict the outcome of a vote. When externalities are present, the outcome of a vote will depend on the expectations of each coalition S regarding the coalition formation of the outsiders when it wishes to veto an alternative. In this way, we have to make assumptions about what a coalition conjectures about outsiders' behavior while defining the core. In this article, we focus on the two extreme cases of pessimistic and optimistic expectations. By (P) and (SO), the worst coalition formation for S is that outsiders form the largest possible coalition which corresponds to $\rho_m = \{S, N \backslash S\}$, and the best coalition formation for S is that each outsider forms a singleton which corresponds to $\rho_s = \{S\} \cup \{\{j\} : j \in N \backslash S\}$. Thus, voting games satisfying (P) and (SO) have positive externalities in the sense that a splitting of the outsiders makes coalition S stronger.

Definition 2 *Let (N, W^e) be a game and take $\alpha \in \{m, s\}$. The core with α-expectations is the set of all alternatives that will not be vetoed by a winning coalition, i.e.*

[2] A preorder $\succeq_i$ over X is continuous if for all $x \in X$, the sets $\{y \in X : x \succeq_i y\}$ and $\{y \in X : y \succeq_i x\}$ are closed relative to X.

[3] A preorder $\succeq_i$ over X is convex if $[\forall x, y, z \in X,\, y \succeq_i x$ and $z \succeq_i x] \Longrightarrow [\lambda y + (1-\lambda) z \succeq_i x]$ for all $\lambda \in [0, 1]$.

$$C^{\alpha}(N, W^e) = \{x \in X : S \in W^e(\rho_\alpha) \Longrightarrow \nexists y \in X : \forall i \in S,\ y \succ_i x\}.$$[4]

The connections between $C^m(N, W^e)$ and $C^s(N, W^e)$ are emphasized by the following result.

Proposition 1 *Let (N, W^e) be a game. Then $C^s(N, W^e) \subseteq C^m(N, W^e)$.*

Proof: Assume that $x \in C^s(N, W^e)$ and by contradiction that $x \notin C^m(N, W^e)$, i.e. there is $S \in W^e(\{S, N \backslash S\})$ and $y \in X$ such that $y \succ_i x$ for each $i \in S$. By (P) and (SO), we know that S is the unique winning coalition for all $\rho \in \Pi(N)$ such that $S \in \rho$. In particular, this is true for ρ_s which contradicts the assumption that $x \in C^s(N, W^e)$. ■

3. The core under pessimistic expectations

In this section, we study the core in the case where the expectations of each coalition are pessimistic. For this purpose, we introduce the associated voting game without externalities (N, W_{pes}) where W_{pes} is the set of winning coalitions facing the largest possible coalition. Formally,

$$W_{pes} = \{S \in S(N) : S \in W^e(\rho_m)\} \tag{3.1}$$

The core of (N, W_{pes}) is defined as the set of all alternatives that will be vetoed by no coalition in W_{pes}, i.e.

$$C(N, W_{pes}) = \{x \in X : S \in W_{pes} \Longrightarrow \nexists y \in X : \forall i \in S,\ y \succ_i x\}.$$

The result below establishes an equivalence between $C^m(N, W^e)$ and $C(N, W_{pes})$.

Fact 1 *Let (N, W^e) be a game. If we consider the transformation from (N, W^e) to (N, W_{pes}) given by (3.1), then $C^m(N, W^e) = C(N, W_{pes})$.*

Proof: The proof follows directly from (3.1). ■

As discussed in the introduction, Nakamura [5] proves that the non-emptiness of the core of a voting game (without externalities) (N, W) is directly related to the combinatorial structure of its winning coalitions.

Definition 3 *The Nakamura number of a voting game (N, W) is the extended natural number $v(N, W)$ defined as follows:*

$$v(N, W) = \begin{cases} \infty & \text{if } \bigcap_{C \in W} C \neq \emptyset; \\ \min\{|W'| : W' \subseteq W \text{ and } \bigcap_{C \in W'} C = \emptyset\} & \text{if } \bigcap_{C \in W} C = \emptyset. \end{cases}$$

[4] The binary relation $\succ_i$ represents the asymmetric component of $\succeq_i$.

The set $\bigcap_{C \in W} C$ is called the collegium of (N, W) and its members the vetoers. If it is empty, then the game (N, W) is non-collegial.

Among the voting games, the quota games have received much attention. A quota game is a voting game (N, W) such that there exists an integer $q \leq n$ satisfying $S \in W$ if and only if $|S| \geq q$. For these games, it has been proved the following result:

Lemma 1 *Let (N, W) be a quota game. Then its Nakamura number is given by $v(N, W) = [n/(n-q)]$.*[5]

For voting games (without externalities), a necessary and sufficient condition for the non-emptiness of the core has been provided by Le Breton [4].

Theorem 1 (Le Breton [4]) *Let (N, W) be a game. Suppose that X is a nonempty, convex and compact subset of $\mathbb{R}^p$, $p \in \mathbb{N}$. Then,*

(i) if $dimX \leq v(N, W) - 2$, then for all $\succeq \in \mathcal{P}_{co}^n$, it holds that $C(N, W) \neq \emptyset$;[6]

(ii) if $dimX \geq v(N, W) - 1$, then there exists $\succeq \in \mathcal{P}_{co}^n$ such that $C(N, W) = \emptyset$.

In order to prove the main theorem of this section, we need the following lemma:

Lemma 2 *For any coalition $S \in S(N)$ and any coalition $T \in S(N)$ such that $S \in W^e(\{S, N \backslash S\})$ and $T \in W^e(\{T, N \backslash T\})$ we have $S \cap T \neq \emptyset$.*

Proof: Suppose by contradiction that $S \cap T = \emptyset$. Since $S \in W^e(\{S, N \backslash S\})$, (SO) implies that $S \in W^e(\{S, T, N \backslash (S \cup T)\})$. By (P), we know that $T \notin W^e(\{S, T, N \backslash (S \cup T)\})$. We conclude from (SO) that $T \notin W^e(\{T, N \backslash T\})$, a contradiction. ■

Lemma 2 establishes that the game (N, W_{pes}) is proper, i.e. $S \in W_{pes}$ and $T \in W_{pes}$ implies that $S \cap T \neq \emptyset$.

Theorem 2 *For any game (N, W^e), it holds that $C^m(N, W^e) \neq \emptyset$ for all $\succeq \in \mathcal{P}_{co}^n$ if and only if $dimX = 1$.*

Proof: ($\Longrightarrow$) Assume that $dimX = 1$. Pick any game (N, W^e) and consider the associated game (N, W_{pes}) as in (3.1). By Lemma 2 we know that $S \cap T \neq \emptyset$ for any coalition $S \in W_{pes}$ and any coalition $T \in W_{pes}$. For the game (N, W_{pes}) this implies that $v(N, W_{pes}) > 2$. Since $dimX = 1$, we conclude by Theorem 1 that $C(N, W_{pes}) \neq \emptyset$ for all $\succeq \in \mathcal{P}_{co}^n$. We conclude from Fact 1 that $C^m(N, W^e) \neq \emptyset$ for all $\succeq \in \mathcal{P}_{co}^n$.
($\Longleftarrow$) Assume that $C^m(N, W^e) \neq \emptyset$ for all $\succeq \in \mathcal{P}_{co}^n$. In order to prove the necessary part of the condition, we have to construct a correspondence W^e (satisfying (P) and (SO)) such that $C^m(N, W^e) \neq \emptyset$ for all $\succeq \in \mathcal{P}_{co}^n$ only if $dimX = 1$. We distinguish four cases:
(a) Assume that $n \geq 6$. For any $\rho \in \Pi(N)$, let us define W^e by

$$W^e(\rho) = \{S \in S(N) : \forall T \in \rho,\ T \neq S,\ |S| > |T|\} \tag{3.2}$$

[5] $[x]$ is the smallest integer greater than or equal to x.
[6] We denote by $dimX$ the dimension of the affine hull of X.

Clearly, W^e satisfies (P) and (SO). Moreover, under pessimistic expectations we have

$$W_{pes} = \{S \in S(N) : |S| \geq n/2 + 1\}.$$

Thus, when W^e is defined by (3.2) the corresponding voting game (N, W_{pes}) is a quota game. By using Lemma 1, some elementary calculus show that $v(N, W_{pes}) = 3$ for any $n \geq 6$.
(b) Assume that $n = 3$. Let us define W^e as follows: $W^e(\{\{1,2\},\{3\}\}) = \{\{1,2\}\}$, $W^e(\{\{1,3\},\{2\}\}) = \{\{1,3\}\}$, $W^e(\{\{2,3\},\{1\}\}) = \{\{2,3\}\}$, and for all other partitions $\rho \in \Pi(N)$, $W^e(\rho) = \emptyset$.
(c) Assume that $n = 4$. Let us define W^e as follows: for all $\rho_{34} \in \Pi(N\backslash\{1,2\})$, $W^e(\{\{1,2\}\} \cup \rho_{34}) = \{\{1,2\}\}$, for all $\rho_{24} \in \Pi(N\backslash\{1,3\})$, $W^e(\{\{1,3\}\} \cup \rho_{24}) = \{\{1,3\}\}$, for all $\rho_{14} \in \Pi(N\backslash\{2,3\})$, $W^e(\{\{2,3\}\} \cup \rho_{14}) = \{\{2,3\}\}$, and for all other partitions $\rho \in \Pi(N)$, $W^e(\rho) = \emptyset$.
(d) Assume that $n = 5$. Let us define W^e as follows: for all $\rho_{345} \in \Pi(N\backslash\{1,2\})$, $W^e(\{\{1,2\}\} \cup \rho_{345}) = \{\{1,2\}\}$, for all $\rho_{245} \in \Pi(N\backslash\{1,3\})$, $W^e(\{\{1,3\}\} \cup \rho_{245}) = \{\{1,3\}\}$, for all $\rho_{145} \in \Pi(N\backslash\{2,3\})$, $W^e(\{\{2,3\}\} \cup \rho_{145}) = \{\{2,3\}\}$, and for all other partitions $\rho \in \Pi(N)$, $W^e(\rho) = \emptyset$.
For any $n \in \{3,4,5\}$, W^e satisfies (P) and (SO). Moreover, we have $W_{pes} = \{\{1,2\},\{1,3\},\{2,3\}\}$, and so $v(N, W_{pes}) = 3$.
In all cases (a), (b), (c) and (d), we conclude from Theorem 1 and Fact 1 that $C^m(N, W^e) \neq \emptyset$ for all $\succeq \in \mathcal{P}^n_{co}$ only if $dimX = 1$. ■

4. The core under optimistic expectations

In this section, we study the core in the case where the expectations of each coalition are optimistic. For this purpose, we introduce the associated voting game without externalities (N, W_{opt}) where W_{opt} is the set of winning coalitions facing singletons. Formally,

$$W_{opt} = \{S \in S(N) : S \in W^e(\rho_s)\} \tag{4.1}$$

The core of (N, W_{opt}) is defined as the set of all alternatives that will be vetoed by no coalition in W_{opt}, i.e.

$$C(N, W_{opt}) = \Big\{x \in X : S \in W_{opt} \Longrightarrow \nexists y \in X \text{ s.t. } \forall i \in S,\ y \succ_i x\Big\}.$$

The result below establishes an equality between $C^s(N, W^e)$ and $C(N, W_{opt})$.

Fact 2 *Let (N, W^e) be a game. If we consider the transformation from (N, W^e) to (N, W_{opt}) given by (4.1) then $C^s(N, W^e) = C(N, W_{opt})$.*

Proof: The proof follows directly from (4.1). ■

Under optimistic expectations, we obtain the opposite result to the one under pessimistic expectations.

Theorem 3 *For any $dimX \in \mathbb{N}$, there exists a game (N, W^e) and a profile of preference relations $\succeq \in \mathcal{P}^n_{co}$ such that $C^s(N, W^e) = \emptyset$.*

Proof: We distinguish two cases:
(a) Assume that $n \geq 4$. Take W^e defined by (3.2). In this case, under optimistic expectations we have

$$W_{opt} = \{S \in S(N) : |S| \geq 2\}.$$

Note that (N, W_{opt}) is a quota game and its Nakamura number is given by $[n/(n-2)] = 2$ for any $n \geq 4$.
(b) Assume that $n = 3$. Let us define W^e as follows: $W^e(\{\{1\},\{2\},\{3\}\}) = \{1\}$, $W^e(\{\{1\},\{2,3\}\}) = \{2,3\}$, and for all other partitions $\rho \in \Pi(N)$, $W^e(\rho) = \emptyset$. In this case, we have $W_{opt} = \{\{1\},\{2,3\}\}$, and so $v(N, W_{opt}) = 2$.
In both cases (a) and (b), we conclude from Theorem 1 and Fact 2 that for any $dimX \in \mathbb{N}$, there exists a game (N, W^e) and a profile of preference relations $\succeq \in \mathcal{P}^n_{co}$ such that $C^s(N, W^e) = \emptyset$. ■

5. Concluding remarks

For the class of voting games with externalities, we have showed that if each coalition has pessimistic expectations, then the core is non-empty if and only if the dimension of the set of alternatives is equal to one. If each coalition has optimistic expectations, we have proved that the non-emptiness of the core is not ensured. Thus, as in Funaki and Yamato [1] in the case of an economy with a common pool resource, whether or not the core is non-empty depends crucially on the expectations of each coalition about outsiders' behavior when it wishes to veto an alternative.
Our core existence results hold for a large class of voting games with externalities since we only impose the two natural properties (P) and (SO) on the correspondence W^e. A natural extension of our work would be to impose other properties on W^e in order to study another class of voting games with externalities.

Acknowledgements

I wish to thank Sylvain Béal, Laurent Mathevet and Philippe Solal for providing numerous suggestions that substantially improved the exposition of the article.

References

1. Funaki, Y.; Yamato T. The core of an economy with a common pool resource: A partition function form approach. *International Journal of Game Theory* **1999**, 28, 157-171.
2. Greenberg J. Consistent majority rules over compact sets of alternatives. *Econometrica* **1979**, 47, 627-636.
3. Hafalir Isa E. Efficiency in coalition games with externalities. *Games and Economic Behavior* **2007**, 61, 242?258.
4. Le Breton M. On the Core of Voting Games. *Social Choice and Welfare* **1987**, 4, 295-305.
5. Nakamura K. The vetoers in a simple game with ordinal preferences. *International Journal of Game Theory* **1979**, 8, 55-61.
6. Schofield N. Social equilibrium and cycles on compact sets. *Journal of Economic Theory* **1984**, 33, 59-71.

7. Strnad J. The structure of continuous-valued neutral monotonic social functions. *Social Choice and Welfare* **1985**, 2, 181-195.
8. Thrall R.M.; Lucas W.F. n-person games in partition function form. *Naval Research Logistic Quarterly* **1963**, 10, 281-298